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University of Washington
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Abstract

In this report a systematic mathematical method is introduced for the solution of problems involving two coupled modes in a coupled system with varying parameters. These problems involve systems of linear differential equations with varying coefficients.

By the use of a linear transformation of the dependent variables and a double diagonalization process, the coupled mode equations are reduced to two decoupled Riccati equations. The final form of the general solution is obtained in terms of four varying coupling coefficients and a transform parameter.

To illustrate some applications of the method, solutions of two special cases which have been solved previously by other workers are obtained. The solutions for a number of special cases, in which the varying coefficients are specified or interrelated, are also obtained. Further possible applications are indicated.

Introduction

During the past decade, workers in different branches of Physics and Engineering have published a considerable amount of material on the application of coupled mode theory to various kinds of coupled systems.[1-8] As far as the mathematical theory is concerned, most researchers have considered only special cases of coupled systems having constant parameters or slowly varying parameters.[5,7] In the latter case approximate solutions have been obtained. Recently, in their study of Faraday rotation in a ferrite rod, Huang and Fan considered the case in which two self-coupling coefficients are equal and one of the mutual coupling coefficients is equal to the negative-conjugate of the other.[3] This report extends the treatment further to more general case of varying parameters, and a systematic mathematical method is introduced to solve the problem of two coupled modes in a coupled system.

In coupled mode theory, the behavior of a coupled system is described in terms of the normal modes of the uncoupled system. The equations that characterize the behavior of two coupled modes takes the form:

$$\frac{da_1}{dz} = A(z)a_1 + B(z)a_2 \tag{1-a}$$

$$\frac{da_2}{dz} = C(z)a_1 + D(z)a_2 \tag{1-b}$$

In these expressions, a_1 and a_2 are mode amplitudes of the coupled system, and $A(z)$, $B(z)$, $C(z)$, and $D(z)$ are mode coupling coefficients which may be any arbitrary functions of z. By the use of a linear transformation of the dependent variables and a double diagonalization process, we are able to obtain general solutions of the coupled mode equations in terms of the four varying parameters A, B, C, D, and a transform parameter h_0, which is a particular solution of a generalized Riccati equation. Since the particular solutions of the generalized Riccati equation have been widely studied, and numerous results

are available in the literature of non-linear differential equations,[9-15] general solutions of equations (1) can be obtained for a large number of special cases. The solutions that we have studied can be applied to many different types of systems. What is of interest here is that the results are of such generality that they allow the description and classification of a wide class of devices in a significant manner.

Further applications of our solutions to the structures of log-periodic antennas, traveling wave antennas, and directional couplers are indicated in the concluding remarks.

I. General Solutions of Coupled-Mode Equations

This section is devoted to solving the two dimensional coupled-mode equations by linear transformations of the dependent variables. In order to simplify the writing, we express the coupled mode equations in matrix form. Let

$$\bar{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad \left[A\right] = \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix} \tag{2}$$

Equations (1) in matrix form become

$$\bar{a}' = \left[A\right]\bar{a} \tag{3}$$

where the prime denotes differentiation with respect to the independent variable z. Now, we introduce the following transformation of the dependent variables in equation (3).[11] Let $\left[R\right]$ be a continuously differentiable, non-singular matrix for $z_1 \leq z \leq z_2$. Under the linear change of variables $\bar{a} \rightarrow \bar{g}$, where

$$\bar{a} = \left[R\right]\bar{g} \tag{4}$$

(3) is transformed into

$$\bar{g}' = \left[R\right]^{-1}\{\left[A\right]\left[R\right] - \left[R\right]'\}\bar{g} \tag{5}$$

For our purpose, we choose $[R]$ specifically to be

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} e^{\int A dz} & 0 \\ 0 & e^{\int D dz} \end{bmatrix} \tag{6}$$

It is easy to show by matrix algebra that

$$\bar{g}' = [S]\,\bar{g} \tag{7}$$

where

$$[S] = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = \begin{bmatrix} 0 & Be^{-\int (A-D)dz} \\ Ce^{\int (A-D)dz} & 0 \end{bmatrix} \tag{8}$$

For a second transformation, let us again assume that $[T]$ be a continuously differentiable non-singular matrix in the same range of z as considered in the first transformation: i.e. $z_1 \leq z \leq z_2$. Under the linear change of dependent variables $\bar{g} \rightarrow \bar{y}$, where

$$\bar{g} = [T]\,\bar{y} \tag{9}$$

(7) is transformed into

$$\bar{y}' = [T]^{-1}\{[S][T] - [T]'\}\,\bar{y} \tag{10}$$

Let us choose $[T]$ specifically to be

$$[T] = \begin{bmatrix} 1 & t_{12} \\ t_{21} & 1 \end{bmatrix} \tag{11}$$

Then (10) becomes after some mathematical manipulations

$$\bar{y}' = [L]\,\bar{y} \tag{12}$$

where

$$[L] = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} = \begin{bmatrix} s_{12}t_{21} & 0 \\ 0 & s_{21}t_{12} \end{bmatrix} \tag{13}$$

t_{12} and t_{21} are transform parameters which are particular solutions of the following differential equations:

$$t_{12}' - s_{12} + s_{21}t_{12}^2 = 0 \tag{14-a}$$

$$t_{21}' - s_{21} + s_{12}t_{21}^2 = 0 \tag{14-b}$$

and s_{12}, s_{21} are the matrix elements given specifically in (8).

Since $\left[L\right]$ in (13) is a diagonal matrix, solutions for the y's can easily be found by solving the decoupled differential equations of (12). They are

$$y_1 = c_1' \, e^{\int s_{12}t_{21}dz} \tag{15-a}$$

$$y_2 = c_2' \, e^{\int s_{21}t_{12}dz} \tag{15-b}$$

where c_1' and c_2' are constants of integration.

According to transformation (9), we have

$$g_1 = y_1 + t_{12}y_2 = c_1' \, e^{\int s_{12}t_{21}dz} + c_2't_{12} \, e^{\int s_{21}t_{12}dz} \tag{16-a}$$

$$g_2 = t_{21}y_1 + y_2 = c_1't_{21} \, e^{\int s_{12}t_{21}dz} + c_2' \, e^{\int s_{21}t_{12}dz} \tag{16-b}$$

Also, from the transformation (4), and the expressions for the g's in (16), we obtain

$$a_1 = c_1'r_{11} \, e^{\int s_{12}t_{21}dz} + c_2'r_{11}t_{12} \, e^{\int s_{21}t_{12}dz} \tag{17-a}$$

$$a_2 = c_1'r_{22}t_{21} \, e^{\int s_{12}t_{21}dz} + c_2'r_{22} \, e^{\int s_{21}t_{12}dz} \tag{17-b}$$

Using the following transformation of the dependent variables in (14),

$$\begin{bmatrix} t_{12} \\ t_{21} \end{bmatrix} = \left[U\right] \begin{bmatrix} h_{12} \\ h_{21} \end{bmatrix} \tag{18}$$

where $\left[U\right]$ is given as

$$U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} e^{-\int(A-D)dz} & 0 \\ 0 & e^{\int(A-D)dz} \end{bmatrix} \tag{19}$$

We then obtain from (14)

$$h'_{12} - (A-D)\, h_{12} + Ch_{12}^2 = B \qquad (20\text{-a})$$

$$h'_{21} + (A-D)\, h_{21} + Bh_{21}^2 = C \qquad (20\text{-b})$$

Also, (17) becomes under the transformation of (19)

$$a_1 = c'_1 r_{11}\, e^{\int s_{12} u_{11} h_{12} dz} + c'_2 r_{11} t_{12}\, e^{\int s_{21} u_{12} h_{21} dz} \qquad (21\text{-a})$$

$$a_2 = c'_1 r_{22} t_{21}\, e^{\int s_{12} u_{22} h_{21} dz} + c'_2 r_{22}\, e^{\int s_{21} u_{11} h_{12} dz} \qquad (21\text{-b})$$

Substituting all the elements of $[R]$, $[S]$, and $[U]$ specified in (6), (8), and (19), we finally obtain the general solutions of the coupled mode equations (1):

$$a_1 = c'_1\, e^{\int (A+Bh_{21}) dz} + c'_2 h_{12}\, e^{\int (D+Ch_{12})\, dz} \qquad (22\text{-a})$$

$$a_2 = c'_1 h_{21}\, e^{\int (A+Bh_{21}) dz} + c'_2\, e^{\int (D+Ch_{12}) dz} \qquad (22\text{-b})$$

where h_{12} and h_{21} are transform parameters which are particular solutions of the generalized Riccati equations (20-a) and (20-b), respectively. It must be noted that in (22) h_{12} and h_{21} should be subject to the condition that $h_{12}h_{21} - 1 \neq 0$, which follows from the non-singular requirement on transform matrix $[T]$.

Realizing that parameters h_{12} and h_{21} are related through the differential equations (20), we can further eliminate one of these parameters. It can be shown that if h_0 is a particular solution of (20-b), $1/h_0$ will be a particular solution of (20-a). In order to satisfy the condition that $h_{12}h_{21} - 1 \neq 0$, $1/h_0$ and h_0 cannot be used as transform parameters for h_{12} and h_{21}, respectively, at the same time.

Let $1/h_0$ be a particular solution of (20-a). Then by Euler's theorem concerning the generalized Riccati equations (see appendix A), we find that

$$h_0 + \frac{\exp \int (D-A-2Bh_0)dz}{\int B \exp \{\int (D-A-2Bh_0)dz\} \cdot dz}$$

is also a particular solution of (20-b). Since the quantity

$$h_{12}h_{21} - 1 \equiv \frac{\exp \int (D-A-2Bh_0)dz}{h_0 \int B \exp \{\int (D-A-2Bh_0)dz\} \cdot dz}$$

does not vanish in the interval $z_1 \leq z \leq z_2$, in general, $1/h_0$ and the above expression are appropriate transform parameters for the system (22). The result of substituting the above particular solutions for the h's in (22-a) is

$$\begin{aligned} a_1 &= c_1' \exp \int \left[A + B\left(h_0 + \frac{\exp \int (D-A-2Bh\ )dz}{\int B \exp\{\int (D-A-2Bh_0)dz\} \cdot dz}\right)\right] \cdot dz \\ &\quad + c_2' \frac{1}{h_0} \exp \int \left(D + \frac{C}{h_0}\right) dz \\ &= c_1' \left[\int B \exp\{\int (D-A-2Bh_0)dz\} \cdot dz \right] \cdot \exp \int (A + Bh_0)dz \\ &\quad + c_2' \frac{1}{h_0} \exp \int \left(D + \frac{C}{h_0}\right) dz \end{aligned} \tag{23}$$

From (20-a), it is found that

$$\frac{1}{h_0} \exp \int \left(D + \frac{C}{h_0}\right)dz = k \exp \int (A+Bh_0)dz \tag{24}$$

where k is an arbitrary constant. Then (23) is reduced to

$$a_1 = \left[c_1 \int B \exp\{\int (D-A-2Bh_0)dz\} \cdot dz + c_2\right] \cdot \exp \int (A+Bh_0)dz \tag{25-a}$$

Similarly, we have

$$\begin{aligned} a_2 = \Big\{ & c_1 h_0 \int B \exp\{\int (D-A-2Bh_0)dz\} \cdot dz + c_1 \exp\{\int (D-A-2Bh_0)dz \\ & + c_2 h_0 \Big\} \exp \int (A+Bh_0)dz \end{aligned} \tag{25-b}$$

where h_0 is a transform parameter which is a particular solution of the well-known generalized Riccati equation (20-b), and c_1, c_2 are constants of integration.

Following the same procedures, we may obtain the solutions in a slightly different form. They are

$$a_1 = \Big\{ c_1 h_0 + c_2 [h_0 \int C \exp\{\int (A-D-2Ch_0) dz\} \cdot dz + \exp\{\int (A-D-2Ch_0) dz\}] \Big\} \cdot \exp \int (D+Ch_0) dz \tag{26-a}$$

$$a_2 = [c_1 + c_2 \int C \exp\{\int (A-D-2Ch_0) dz\} \cdot dz] \exp \int (D+Ch_0) dz \tag{26-b}$$

where h_0 is a particular solution of the generalized Riccati equation (20-a), and c_1, c_2 are constants of integration.

Our general solutions of the coupled-mode equations have thus been obtained by three successive linear transformations of dependent variables and a double diagonalization process. There is a transform parameter involved in our solutions, which is a particular solution of a generalized Riccati equation. The problem of finding general solutions of the coupled-mode equations is now reduced to that of finding a particular solution of a generalized Riccati equation.

The next section will be concerned with a discussion of the generalized Riccati differential equation. A method of obtaining particular solutions of the generalized Riccati equation is given in Appendix B.

II. The Generalized Riccati Equation

It is customary to give the name "generalized Riccati equation" to any equation of the form

$$\frac{dy}{dz} + Py + Qy^2 = R \tag{27}$$

where P, Q, and R are given functions of z.

This equation has considerable theoretical importance, since its solutions are free from movable branch points and can have only movable poles.[10] It is a special case of the Abel equation. It is supposed that neither R nor Q is identically zero. If Q = 0, the equation is linear; if R = 0, the equation is reducible to the linear form by taking 1/y as a new variable.

It has been shown by Euler that, if a particular solution of the generalized Riccati equation is known, the general solution can be obtained by two quadratures; if two particular solutions are known, the general solution is obtainable by a single quadrature. And it follows from theorems by Weyr and Picard that, if three particular solutions are known, the general solution can be effected without a quadrature.

The equation (27) is easily reduced to a linear equation of the second order by taking a new dependent variable u defined by the equation

$$y = \frac{1}{Q} \frac{d\log u}{dz} \tag{28}$$

The equation then becomes

$$\frac{d^2u}{dz^2} + \left(P - \frac{1}{Q} \frac{dQ}{dz}\right) \frac{du}{dz} - Q R u = 0 \tag{29}$$

Conversely, if in the general linear equation of the second order

$$P_0 \frac{d^2u}{dz^2} + P_1 \frac{du}{dz} + P_2 u = 0 \tag{30}$$

(where P_0, P_1, P_2 are given functions of z), one may write

$$u = e^{\int Qydz} \tag{31}$$

From which the equation defining y becomes

$$\frac{dy}{dz} + \left(\frac{P_1}{P_0} + \frac{1}{Q} \frac{dQ}{dz}\right) y + Qy^2 = -\frac{P_2}{P_0 Q} \tag{32}$$

which is of the same type as (27). The complete equivalence of the generalized Riccati equation with linear equation of the second order is consequently established.

In the following sections, we will apply our method to a number of special cases for which solutions have been previously established by other workers.

III. Uniform Coupling of Two Lossless Modes of Propagation

Consider two waves with time dependence $e^{j\omega t}$ which are weakly coupled. It can be shown that the coupled-mode equations for this situation are

$$\frac{da_1}{dz} = -j\beta_1 a_1 + c_{12} a_2 \tag{33-a}$$

$$\frac{da_2}{dz} = c_{21} a_1 - j\beta_2 a_2 \tag{33-b}$$

The c_{12} and c_{21} are the mutual coupling coefficients per unit length. The coupling is assumed uniform over the length of the coupler, so that c_{12} and c_{21} are independent of z. The modes are assumed lossless. For this case, the matrix elements of $[A]$ are constants. Let

$$[A] = \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix} = \begin{bmatrix} -j\beta_1 & C_{12} \\ c_{21} & -j\beta_2 \end{bmatrix} \tag{34}$$

where β_1, β_2, c_{12}, and c_{21} are independent of z.

From (34), (20-b) becomes

$$h' - j(\beta_1 - \beta_2)\, h + c_{12} h^2 = c_{21} \tag{35}$$

A particular solution of (35) can easily be found to be

$$h_0 = -\frac{\beta_b}{c_{12}} \tan \beta_b z + j\,\frac{\beta_d}{c_{12}} \tag{36}$$

where

$$\beta_d = \frac{\beta_1 - \beta_2}{2} \tag{37}$$

and

$$\beta_b = \sqrt{\beta_d^{\,2} + c_{12} c_{21}} \tag{38}$$

Substituting (36) in (25) leads to

$$a_1 = \left[c_1 \left(\frac{-c_{12}}{\beta_b} \right) \sin \beta_b z + c_2 \cos \beta_b z \right] e^{-j\beta_a z} \tag{39-a}$$

$$a_2 = \left[\left(c_1 + c_2 \frac{j\beta_d}{c_{12}} \right) \sin \beta_b z - \left(c_1 \frac{j\beta_d}{\beta_b} + c_2 \frac{\beta_b}{c_{12}} \right) \cos \beta_b z \right] e^{-j\beta_a z} \tag{39-b}$$

where

$$\beta_a = \frac{\beta_1 + \beta_2}{2} \tag{40}$$

Equations (39) can be rewritten in a different form

$$a_1 = A_1 e^{-j\gamma_1 z} + A_2 e^{-j\gamma_2 z} \tag{41-a}$$

$$a_2 = \frac{1}{jc_{12}} \left[(\gamma_1 + \beta_1) A_1 e^{-j\gamma_1 z} + (\gamma_2 + \beta_1) A_2 e^{-j\gamma_2 z} \right] \tag{41-b}$$

where

$$\gamma_1 = \beta_a + \beta_b \tag{42-a}$$

and

$$\gamma_2 = \beta_a - \beta_b \tag{42-b}$$

Here γ_1 and γ_2 are normal mode propagation constants and A_1, A_2 are arbitrary constants.

These solutions agree with those obtianed by other methods.

IV. Coupled-Mode Description of Guided Wave Propagation through a Ferrite Rod

A general set of coupled mode equations for wave propagation in a waveguide partially filled with a gyromagnetic medium has been derived by Huang and Fan. They are

$$\frac{da_e}{dz} = - K_{ee} a_e - K_{eo} a_o \tag{43-a}$$

$$\frac{da_o}{dz} = - K_{oe} a_e - K_{oo} a_o \tag{43-b}$$

Also

$$K_{ee}(z) = K_{oo}(z) \tag{44-a}$$

$$K_{eo}(z) = - K_{oe}(z) \tag{44-b}$$

where the K's are varying mode coupling coefficients.

In order to solve this set of equations, we write

$$[A] = \begin{bmatrix} A(z) & B(z) \\ \\ C(z) & D(z) \end{bmatrix} = \begin{bmatrix} -K_{ee}(z) & K_{eo}(z) \\ \\ K_{eo}(z) & -K_{ee}(z) \end{bmatrix} \tag{45}$$

Then (20-b) becomes

$$h' - K_{eo}(h^2 + 1) = 0 \tag{46}$$

It is easy to show that a particular solution of (46) is

$$h_0 = \tan \int K_{eo}\, dz \tag{47}$$

Substituting h_o in (25), we obtain the general solutions of (43):

$$\begin{aligned} a_1 &= \left[c_1 \int (-K_{eo}) \exp\{\int 2K_{eo} \tan(\int K_{eo} dz)\, dz\} \cdot dz + c_2\right] \\ &\quad \cdot \exp\left[- \int \{K_{ee} - K_{eo} \tan(\int K_{eo}\, dz)\} \cdot dz\right] \\ &= \left[- c_1 \int K_{eo} \sec^2(\int K_{eo} dz) dz + c_2\right] \cdot \cos(\int K_{eo} dz) \cdot \exp(-\int K_{ee} dz) \\ &= \left[- c_1 \tan(\int K_{eo} dz) + c_2\right] \cdot \cos(\int K_{eo} dz) \cdot \exp(-\int K_{ee} dz) \\ &= \left[- c_1 \sin(\int K_{eo} dz) + c_2 \cos(\int K_{eo} dz)\right] \cdot \exp(-\int K_{ee} dz) \\ &= \frac{d_1}{\sqrt{2}}\, e^{-\int (K_{ee} + jK_{eo}) dz} + \frac{d_2}{\sqrt{2}}\, e^{-\int (K_{ee} - jK_{eo}) dz} \end{aligned} \tag{48-a}$$

$$a_2 = \left[- c_1 \tan^2(\int K_{eo} dz) + c_2 \sec^2(\int K_{eo} dz) + c_2 \tan(\int K_{eo} dz) \right] \cdot \cos(\int K_{eo} dz) \cdot \exp(-\int K_{ee} dz)$$

$$= \left[c_1 \cos(\int K_{eo} dz) + c_2 \sin(\int K_{eo} dz) \right] \cdot \exp(-\int K_{ee} dz)$$

$$= j \left[\frac{d_1}{\sqrt{2}} e^{-\int (k_{ee} + jK_{eo}) dz} - \frac{d_2}{\sqrt{2}} e^{-\int (K_{ee} - jK_{eo}) dz} \right] \tag{48-b}$$

where d_1 and d_2 are arbitrary constants. These expressions agree with the solutions given in Reference 3.

V. Case (i)

Here we consider the situation in which

$$A(z) = D(z) \tag{49-a}$$

and

$$B(z) = C(z). \tag{49-b}$$

For this special case, the generalized Riccati equation (20-b) becomes

$$h' + B(h^2-1) = 0 \tag{50}$$

A particular solution of (50) is

$$h_0 = \coth \int B dz \tag{51}$$

The solutions which follow are then

$$a_1 = \left[c_1 \int B \exp\{ \int - 2B \coth(\int B dz) dz\} \cdot dz + c_2 \right] \cdot \exp \int \{A + B \coth(\int B dz)\} \cdot dz$$

$$= \left[c_1 \int B \operatorname{csch}^2(\int B dz) dz + c_2 \right] \cdot (\sinh \int B dz) \cdot \exp(\int A dz)$$

$$= \left(- c_1 \cosh \int B dz + c_2 \sinh \int B dz \right) \exp\left(\int A dz \right)$$

$$= d_1 \, e^{\int (A+B) dz} + d_2 \, e^{\int (A-B) dz} \qquad (52\text{-a})$$

$$a_2 = \left\{ c_1 \coth^2\left(\int B dz \right) + c_1 \operatorname{sech}^2\left(\int B dz \right) + c_2 \coth\left(\int B dz \right) \right\}$$

$$\cdot \left(\sinh \int B dz \right) \cdot \exp\left(\int A dz \right)$$

$$= - d_2 \, e^{\int (A+B) dz} + d_2 \, e^{\int (A-B) dz} \qquad (52\text{-b})$$

If $B(z)$ and $C(z)$ are purely imaginary, then $B(z) = - C^*(z)$. This situation represents a passive coupling mechanism in the coupled-mode of propagation.

VI. Case (ii)

Case (ii) deals with the situation in which

$$A(z) = j \left[\beta(z) + \mu_1 \eta(z) \right] \qquad (53\text{-a})$$

$$D(z) = j \left[\beta(z) - \mu_2 \eta(z) \right] \qquad (53\text{-b})$$

And

$$B(z) = C(z) = j \mu_3 \eta(z) \qquad (53\text{-c})$$

where $\beta(z)$ and $\eta(z)$ are any arbitrary function of z and μ_1, μ_2, μ_3 are arbitrary constants. Substitution of (53) in (20-b) yields

$$h' + j(\mu_1 + \mu_2) \, \eta(z) \, h + j\mu_3 \, \eta(z) \, h^2 = j\mu_3 \, \eta(z) \qquad (54)$$

A particular solution of (54) is

$$h_0 = j \sqrt{1 + \left(\frac{\nu}{2}\right)^2} \, \tan\left(\mu \sqrt{1 + \left(\frac{\nu}{2}\right)^2} \int \eta(z) dz\right) - \frac{\nu}{2} \qquad (55)$$

The results obtained by substituting of (55) into (25) are

$$a_1 = \left[c_1\cos\lambda \int \eta(z)dz + c_2\{j \sin\lambda \int \eta(z)dz + \frac{\nu}{2} \cos\lambda \int \eta(z)dz\}\right]$$

$$\cdot \exp j \int \{\beta(z) + \frac{\mu_1-\mu_2}{2} \eta(z)\} \cdot dz \tag{56-a}$$

$$a_2 = \left[c_1 \ j\{\sin\lambda \int \eta(z)dz - \frac{\nu}{2} \cos\lambda \int \eta(z)dz\}\right.$$

$$\left. + c_2\cos\lambda \int \eta(z)dz \right] \cdot \exp j \int \{\beta(z) + \frac{\mu_1-\mu_2}{2} \beta(z)\} \cdot dz \tag{56-b}$$

where

$$\lambda = \sqrt{\mu_3^2 + (\frac{\mu_1+\mu_2}{2})^2} \tag{57}$$

When μ_3 is real, we have the condition $B(z) = - C^*(z)$. This is the passive coupling case in which the group velocities of the waves are in the same direction.

VII. Case (iii)

Here we consider A, B, C, and D to be related in the following manner:

$$A(z) = j\left[\beta(z) + \mu_1\eta(z)\right] \tag{58-a}$$

$$D(z) = j\left[\beta(z) - \mu_2\eta(z)\right] \tag{58-b}$$

$$C(z) = - B(z) = j\mu_3\eta(z) \tag{58-c}$$

For this situation, the particular solution of (20-b) can be found to be

$$h_0 = - j \sqrt{(\frac{\nu}{2})^2-1} \ \tanh \mu_3 \sqrt{(\frac{\nu}{2})^2-1} \int \eta(z)dz + \frac{\nu}{2} \tag{59}$$

The general solutions for a_1 and a_2 are then

$$a_1 = \Big[c_1 \cosh \kappa \int \eta(z)dz + c_2\{j\sinh \kappa \int \eta(z)dz + \frac{\nu}{2} \cosh \kappa \int \eta(z)dz\}$$

$$\cdot \exp j(\mu_1+\mu_2) \int \eta(z)dz\Big] \cdot \exp j \int \{\beta(z) + \frac{\mu_1-\mu_2}{2} \quad \eta(z)\}dz \qquad (60\text{-a})$$

$$a_2 = \Big[c_1\{j\sinh \kappa \int \eta(z)dz - \frac{\nu}{2} \cosh \kappa \int \eta(z)dz$$

$$+ c_2\{\cosh \kappa \int \eta(z)dz\} \cdot \exp j(\mu_1+\mu_2) \int \eta(z)dz\Big]$$

$$\cdot \exp j \int \{\beta(z) + \frac{\mu_1-\mu_2}{2} \quad \eta(z)\}dz \qquad (60\text{-b})$$

where

$$\kappa = \sqrt{(\frac{\mu_1+\mu_2}{2})^2 - \mu_3^2} \qquad (61\text{-a})$$

$$\nu = \frac{\mu_1+\mu_2}{\mu_3} \qquad (61\text{-b})$$

If C(z) and B(z) are purely imaginary, $C(z) = B^*(z)$. This is the active coupling case in which the group velocities for the two waves are in opposite directions.

VIII. Case (iv)

In this case we consider the relations

$$A(z) \simeq D(z) \qquad (62\text{-a})$$

$$B(z) = \pm C(z) \qquad (62\text{-b})$$

The approximation (62-a) is a fundamental assumption commonly made when treating a pair of weakly coupled modes. Under this assumption, we will neglect the second term, which is small compared with others, in the generalized Riccati equation.

Thus (20-b) becomes

$$h' + B(z)\, h^2 = \pm\, B(z) \tag{63}$$

$$h_0 = \tan \int B dz \tag{64}$$

The expression (64) is a particular solution of (63) taking the positive sign, and

$$h_0 = \coth \int B dz \tag{65}$$

is a particular solution of (63) corresponding to the negative sign. On substitution of these particular solutions in (22), we obtain approximate solutions for a_1 and a_2.

With the upper sign in (62-b), we have

$$a_1 \approx \left\{ -c_1 \left[\int \{B \exp \int (D-A)dz\} \cdot dz \right] \cosh \int B dz + c_2 \sinh \int B dz \right\} \cdot \exp \int A\, dz \tag{66-a}$$

$$a_2 \approx \left\{ c_1 \left\{ \left[\int \{B \exp \int (D-A)dz\} \cdot dz \right] \coth^2\left(\int B dz\right) + \exp \int (A-D)dz \cdot \operatorname{sech}^2 \left(\int B dz\right) \right\} \cdot \sinh \int B dz + c_2 \cosh \int B dz \right\} \exp \int A dz \tag{66-b}$$

With the lower sign in (62-b), we have

$$a_1 \approx \left\{ c_1 \left[\int \{B \exp \int (D-A)dz\} \cdot dz \right] \sin \int B dz + c_2 \cos \int B dz \right\} \cdot \exp \int A dz \tag{67-a}$$

$$a_2 \approx \left\{ c_1 \left\{ - \left[\int \{ B \exp \int (D-A)dz\} \cdot dz \right] \tan^2 \int B dz + \exp \int (A-D)dz \cdot \sec^2 \int B dz \right\} \cdot \cos \int B dz - c_2 \sin \int B dz \right\} \cdot \exp \int A dz \tag{67-b}$$

IX. Concluding Remarks

A systematic method has been introduced to obtain solutions of the two dimensional coupled-mode equations with varying coefficients. Some special cases for which solutions are already known have been given to illustrate the application of our method. The solutions of sections V-VIII may be applied to physical problems: e.g. directional couplers, microwave antennas, etc.

The coupled-mode theory of propagation has been applied by R. A. Sigelmann and D. K. Reynolds in their study of a traveling wave antenna with broad band-width.[8] Because of the systematic method of solving the coupled-mode equations developed here, it may be possible to find solutions for the log-periodic structure by the coupled mode approach.

It is also anticipated that the coupled mode theory as applied to propagation problems will have further application in the design of broadband directional couplers in which the coupling mechanism is of the log-periodic type.

Appendix A

Euler's Theorem Concerning the Generalized Riccati Equation:

It has been shown by Euler that, if a particular solution of the generalized Riccati equation is known, the general solution can be obtained by two quadratures.

To prove the result, let y_0 be a particular solution of

$$\frac{dy}{dz} + Py + Qy^2 = R \tag{A-1}$$

and write $y = y_0 + \frac{1}{v}$. The equation in v is

$$\frac{dv}{dz} - (P + 2Qy_0)\, v + Q = 0 \tag{A-2}$$

of which the solution is

$$v \exp\{-\int(P + 2Qy_0)dz\} - \int Q \exp\{-\int(P + 2Qy_0)dz\} \cdot dz = 0 \tag{A-3}$$

and, since $b = \frac{1}{(y-y_0)}$, the truth of the theorem is manifest.

Appendix B

Particular Solutions of the Generalized Riccati Equation

Sometimes one or more particular solutions can be found by inspection or by a lucky guess. More formal procedures apply in dertain cases.

I. The form of the generalized Riccati equation is

$$y' = f(x) + g(x)y + h(x)y^2 \tag{B-1}$$

Let $y = u(x)/h(x)$ and (B-1) is converted into

$$u' = F(x) + G(x)u + u^2 \tag{B-2}$$

where $F(x) = f(x)h(x)$; $G(x) = g(x) + \frac{h'}{h}$. If both $F(x)$ and $G(x)$ are polynomials. See Sections (a), (b), (c) in turn. If $f(x)$, $g(x)$, and $h(x)$ are polynomials, it will not necessarily be true that $G(x)$ is a polynomial, hence the methods of those sections may not apply; see, however, Section (d) for a possible procedure.

(a) The equation becomes

$$u'(x) = F(x) + u^2 \tag{B-3}$$

and $F(x)$ is a polynomial with $G(x) = 0$. There are two possibilities.

(i) The degree of $F(x)$ is odd. There is no polynomial solution of (B-3); hence, none of (B-1).

(ii) The degree of $F(x)$ is even. Two possible polynomial solutions may exist. To find them, first note that if $P(x)$ were a polynomial of even degree $2n$, then $\sqrt{P(x)}$ could be expanded in series of the form.

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots\cdot + a_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \cdots\cdot$$

Perform such an expansion on $\sqrt{-F(x)}$ but stop the calculation with the constant term. Call this result $X(x)$; it is a polynomial of degree n, if $F(x)$ is of degree $2n$. The coefficients in the

polynomial could be found by a simple modification of the square-root extraction method of elementary algebra, by the undetermined coefficients, or by expansion in a Maclaurin series.

If the differential equation has special polynomial solutions, they are given by

$$u = \pm X(x)$$

Test both of them, for both, one, or neither may satisfy the differential equation. If neither is a solution of (B-3), there are no polynomial solutions of (B-3) and hence none of (B-1).

(b) $G(x) \neq 0$. Let $u = w(x) - \frac{G(x)}{2}$ and (B-2) becomes $w^2(x) = H(x) + w$ where $4H(x) + G^2(x) = 4F(x) + 2G'(x)$. The equation in $w(x)$ is similar to (B-3); hence, with slight modifications the procedure is very much like that in (a). Calculate

$$Q(x) = G^2 - 4F - 2G'$$

There are two cases.

(i) The degree of $Q(x)$ is odd. There is no polynomial solution of either (b-2) or (B-1).

(ii) The degree of $Q(x)$ is even. Expand $\sqrt{Q(x)}$ as in (a) again, stopping with constant term and call the resulting polynomial $X(x)$. There are two possible polynomial solutions of (B-2)

$$u = -\frac{(G \pm X)}{2}$$

Test both, for neither may satisfy the differential equation, see (c).

(c) $Q(x)$ is a constant. This is the necessary and sufficient condition that both solutions of (b) satisfy the differential equation. There are two cases.

(i) $X(x) = k \neq 0$. Introduce a new variable with the relation

$$u = -\frac{(G+k)}{2} + \frac{1}{v}$$

so that (B-2) becomes

$$v'(x) = 1 + kv$$

It is separable with solution $v = ce^{kx} + \frac{1}{k}$ and the corresponding solution of (B-2) is

$$u = -\frac{(G+k)}{2} + (ce^{kx} + \frac{1}{k})^{-1}$$

Two special polynomial solutions of it result with $c = 0, \infty$.

(ii) $X(x) = k = 0$. The equation in $v(x)$ becomes $v'(x) + 1 = 0$, which is separable, with solution $v + x = c$. The solution of (B-2) is

$$u = -\frac{G}{2} - \frac{1}{c-x}$$

(d) Polynomial coefficients:

Suppose that (B-1) has the form

$$\phi(x)y' = f(x) + g(x)y + h(x)y^2 \qquad \text{(B-4)}$$

where all of the coefficients are polynomials. Assume it to have a special polynomial solution $y_1 = R(x)$ and use a new variable $y = u + y_1$ so that it becomes

$$\phi(x)u' = F(x)u + h(x)u^2$$

where $F(x) = g(x) + 2h(x)R(x)$. If there is a polynomial solution of (B-4), having the form $u = (x-a)^m$, $m \geq 1$ then $\phi(x)$ will also contain a factor $(x-a)$. This property restricts the possible types of polynomial solutions. Unlike the previous cases, where only two possible polynomials may exist, the more general solution can have a larger number of such solutions. See (e) for further discussion of this case.

(e) Several polynomial solutions

Given (B-4), any specified number of polynomial solutions can be constructed. The general solution can be taken as

$$y = \frac{cf_1(x) + f_2(x)}{cf_3(x) + f_4(x)} .$$

If $f_1f_4 - f_2f_3 \neq 0$, then (B-1) can be written in the form of (B-4) with $\phi(x) = f_1f_4 - f_2f_3$; $f(x) = f_1f_2' - f_1'f_2$; $g(x) = f_2f_3' - f_2f_3 + f_4f_1' - f_1f_4'$; $h(x) = f_3f_4' - f_4f_3'$.

Now suppose that the $f_i(x)$ are polynomials. Select $f_1(x) = 0$; choose any polynomials desired as $f_3(x)$ and $f_4(x)$. Assign n special values $c_1, c_2, \cdots, c_n$ to the arbitrary constant and require that

$$f_2(x) = (c_1f_3 + f_4)\,(c_2f_3 + f_4) \cdots\cdot (c_nf_3 + f_4)$$

The differential equation (B-4) will have (n+1) polynomial solutions,

$$y = 0 \quad \text{and} \quad y_i = \frac{f_2}{c_if_3+f_4} \qquad i = 1, 2, \cdots, n$$

The special solution $y = 0$ can be avoided and (n+1) special solutions retained if the variables are transformed with the relation $u(x) = g(x) + P(x)$ where $P(x)$ is any polynomial.

The considerations of this section could be very helpful in constructing an equation with a predetermined number of polynomial solutions.

II. The following properties of the generalized Riccati equation sometimes apply in attempting to solve it.

(a) Removal of the linear term

The result, which is similar to (B-3) can be achieved in three different ways. It might then be treated as in I, but see also (b) for a special case.

(i) Let $y = u(x)e^{\phi(x)}$, $\phi(x) = \int g(x)dx$ and (B-1) becomes

$$u'(x) = F(x) + G(x)u^2 \tag{B-5}$$

where $F(x) = f(x)e^{-\phi(x)}$, $G(x) = h(x)e^{\phi(x)}$.

(ii) Let $y = u - v(x)$, $v(x) = \frac{g(x)}{2h(x)}$. If g, h are differentiable and $h(x) \neq 0$, the result is again like (B-5) but now with

$$F(x) = f + v' - \frac{g^2}{4h} ; \quad G(x) = h(x) .$$

(iii) Let $y = u(z)\, e^{\phi(x)}$; $\phi(x) = \int g(x)dx$; $z = - \int he^{\phi}dx$. In this case (B-1) becomes

$$u'(z) = F(z) - u^2(z) \tag{B-6}$$

with $F(z)h(z) = - f(z)\, e^{-2\phi}$

(b) Relation between the coefficients when certain relations exist between the coefficient of (B-1) its solution may be easy to obtain.

(i) Look for two constants a, b with $|a| + |b| > 0$ so that

$$a^2f + abg + b^2h = 0$$

If $a = 0$, then $f(x) = 0$ and the equation is linear. If $a \neq 0$, then $y_1 = b/a$ is a particular solution of (B-1). A simple case arises if $f + g + h = 0$, for then $a = b = 1$ and $y_1 = 1$.

(ii) Use (a) to remove the linear term in (B-1). Then if F(x) is proportional to G(x) in (B-5), the result is separable. In the original variable and functions the requirement is $f(x) = A^2h(x)\int \exp(2\, g(x)dx)$ where A is a constant for proportionality. The solution of (B-1) is then

$$y = \sqrt{\frac{f}{h}} \tan(\int \sqrt{fh}\, dx + C); \qquad fh > 0$$

if $fh < 0$, replace tan by tanh and insert a minus sign under both radicals.

(iii) Assume that a special solution of (B-1) exists so that

$$2hy_1 = X(x) - g(x)$$

where X(x) is determined from the relation

$$f(x) = hy_1{}^2 - X(x)y_1 + y_1{}'$$

The last equation imposes a severe restriction on the form of f(x) but if it is satisfied, y_1 can be found. A few cases that might be tested are

$X = 0,\ 4f = g^2/h - 2(g/h)'$;

$X = - h'/h,\quad 4f = 2(\frac{1}{h})'' - 2(g/h)' - h(1/h)'^2 + g^2/h$;

$X = g - 2\sqrt{fh},\ 2g = 4\sqrt{fh} + f'/f - f'/h$.

III. If all the tests in I and II fail, transform the equation into one of second order. Change the dependent variable by the transformation

$$yh(x)\ u(x) + u'(x) = 0;\quad \ln U(x) + \int yh(x)dx = 0$$

The result i

$$u''(x) + P(x)u' + Q(x)u = 0$$

where $p(x) = - (g + h'/h)$, $Q(x) = f(x)h(x)$. It is linear and of second order. Such equations have been studied extensively and this may be the most suitable procedure unless the given first-order equation has some special property.

IV. There are fourteen transforms listed below in pairs for solving a certain class of Riccati's equation.[16] The intermediate steps dealing with transforms and solutions of these equations are avoided. However, it is appropriate to cite the first case of the transforms listed below to explain the intermediate steps omitted. Take transform 1,

$$y = z/(Pz + Qz'). \tag{B-7}$$

Substitute this into a generalized Riccati's equation,

$$y' = \alpha(x)y + \beta(x)y^2 + \gamma(x) \qquad \text{(B-8)}$$

where α, β and γ are undetermined variable coefficients and only α can become zero, typical of Riccati's class of equations.

Substitute (B-8) into (B-7)

$$Qz'^2 - P'z^2 - (Q'z' + Qz'')z + (Pz + Qz')\alpha z + \beta z^2$$

$$= (Pz + Qz' + 2PQzz') \qquad \text{(B-9)}$$

There are four places where z^2 appears on the other hand, there are two places where z'^2 appears. The key idea is to eliminate terms z^2 and z'^2 in (B-9), thus degenerate (B-9) into a first-order linear differential equation. Because z'^2 appears in two places, the unique choice for $\gamma(x)$ is made first,

$$\gamma(x) = \frac{1}{Q(x)} \qquad \text{(B-10)}$$

The coefficients for z^2 are collected to specify $\alpha(x)$ and $\beta(x)$,

$$- P' + \alpha P + \beta - \frac{P^2}{Q} = 0 \ . \qquad \text{(B-11)}$$

Note that there are three ways of specifying while retaining (B-8) in Riccati's class of equations. If

$$\alpha = 0 \qquad \text{(B-12)}$$

then

$$\beta = P' + \frac{P^2}{Q} \ . \qquad \text{(B-13)}$$

If

$$\alpha = \frac{P}{Q} \qquad \text{(B-14)}$$

then

$$\beta' = P' \ . \qquad \text{(B-15)}$$

If

$$\alpha = (\log_e P)' \qquad \text{(B-16)}$$

then

$$\beta = \frac{P^2}{Q} \tag{B-17}$$

Judging from these developments, it is quite obvious that a transform of type

$$y = \frac{z'}{Pz + Qz'} \tag{B-18}$$

does not solve a Riccati's equation. If (B-18) is substituted in (B-8), there is only one place where z^2 appears. This forces P to zero to eliminate the term of z^2. If P is zero in (B-18), y is not related to another variable z. Following the above steps, the transforms listed below are obtained and may be applied for exact solutions for a certain class of Riccati's non-linear differential equations. Only fourteen transforms in pairs are listed below but it is possible to find many more transforms in pairs.

Transform 1:

$$y = \frac{z}{Pz + Qz'} \tag{B-19}$$

solves

$$y' + (P' + \frac{P^2}{Q})y = \frac{1}{Q} \tag{B-20}$$

$$y' + \frac{Py}{Q} + P'y^2 = \frac{1}{Q} \tag{B-21}$$

$$y' + (\log_e P)'y + \frac{P^2y^2}{Q} = \frac{1}{Q} \tag{B-22}$$

Transform 2:

$$y = \frac{(Pz + Qz')}{z} \tag{B-23}$$

solves

$$y' + \frac{y^2}{Q} = P' + \frac{P^2}{Q} \tag{B-24}$$

$$y' - \frac{Py}{Q} + \frac{y^2}{Q} = P' \tag{B-25}$$

$$y' - (\log_e P)'y + \frac{y^2}{Q} = \frac{P^2}{Q} \tag{B-26}$$

Transform 3:

$$y = \frac{z}{z + PQz'} \tag{B-27}$$

solves

$$y' - [\beta(x) - \frac{1}{PQ}]\, y + \beta(x)y^2 = \frac{1}{PQ} \tag{B-28}$$

where $\beta(x)$ is any arbitrary function of x.

Transform 4:

$$y = \frac{(z + PQz')}{z} \tag{B-29}$$

solves

$$y' + [\gamma(x) - \frac{1}{PQ}]\, y + \frac{y^2}{PQ} = \gamma(x) \tag{B-30}$$

where $\gamma(x)$ is any arbitrary function of x.

Transform 5:

$$y = \frac{PQz}{(Pz + Qz')} \tag{B-31}$$

solves

$$y' + \frac{(P - Q')y^2}{Q^2} = P \tag{B-32}$$

$$y' + \frac{Py}{Q} + (Q^{-1})'y^2 = P \tag{B-33}$$

$$y' - (\log_e Q)'y + \frac{Py^2}{Q^2} = P \tag{B-34}$$

Transform 6:

$$y = \frac{(Pz + Qz')}{PQz} \tag{B-35}$$

solves

$$y' + Py^2 = \frac{(P-Q')}{Q^2} \tag{B-36}$$

$$y' - \frac{Py}{Q} + Py^2 = (Q^{-1})' \tag{B-37}$$

$$y' + (\log_e Q)'y + Py^2 = \frac{P}{Q^2} \tag{B-38}$$

Transform 7:

$$y = \frac{z}{(PQz + z')} \tag{B-39}$$

solves

$$y' + [(PQ)^2 + (PQ)']y^2 = 1 \tag{B-40}$$

$$y' + PQy + (PQ)'y^2 = 1 \tag{B41}$$

$$y' + [\log_e PQ]'y + (PQ)^2y^2 = 1 \tag{B-42}$$

Transform 8:

$$y = \frac{(PQz + z')}{z} \tag{B43}$$

solves

$$y' + y^2 = (PQ)^2 + (PQ)' \tag{B-44}$$

$$y' - PQy + y^2 = (PQ)' \tag{B-45}$$

$$y' - [\log_e PQ]' + y^2 = (PQ)^2 \tag{B-46}$$

Transform 9:

$$y = \frac{Pz}{(Pz + Qz')} \tag{B-47}$$

solves

$$y' - [\beta(x) - \frac{P}{Q}]\, y + \beta(x)\, y^2 = \frac{P}{Q} \tag{B-48}$$

where $\beta(x)$ is any arbitrary function of x.

Transform 10:

$$y = \frac{(Pz + Qz')}{Pz} \tag{B-49}$$

solves

$$y' + [\gamma(x) - \frac{P}{Q}]\, y + \frac{Py^2}{Q} = \gamma(x) \tag{B-50}$$

where $\gamma(x)$ is any arbitrary function of x.

Transform 11:

$$y = \frac{Qz}{(Pz + Qz')} \tag{B-51}$$

solves

$$y' + \left\{[\log_e (\frac{P}{Q})]' + \frac{P}{Q} - \frac{Q\ \beta(x)}{P}\right\} y + \beta(x)y^2 = 1 \tag{B-52}$$

where $\beta(x)$ is any arbitrary function of x.

Transform 12:

$$y = \frac{(Pz + Qz')}{Qz} \tag{B-53}$$

solves

$$y' - \left\{\frac{P}{Q} + [\log_e \frac{P}{Q}]' - \frac{Q\ \gamma(x)}{P}\right\} y + y^2 = \ (x) \tag{B-54}$$

where $\gamma(x)$ is any arbitrary function of x.

Transform 13:

$$y = \frac{PQz}{(z + z')} \tag{B-55}$$

solves

$$y' + [\frac{1}{PQ} + (\frac{1}{PQ})'] y^2 = PQ \tag{B-56}$$

$$y' + y + (\frac{1}{PQ})' y^2 = PQ \tag{B-57}$$

$$y' - (\log_e PQ)' y + \frac{y^2}{PQ} = PQ \tag{B-58}$$

Transform 14:

$$y = \frac{(z + z')}{PQz} \tag{B-59}$$

solves

$$y' + PQy^2 = [\frac{1}{PQ} + (\frac{1}{PQ})'] \tag{B-60}$$

$$y' - y + PQy^2 = (\frac{1}{PQ})' \tag{B-61}$$

$$y' + (\log_e PQ)'y + PQy^2 = \frac{1}{PQ} \tag{B-62}$$

V. A Table of Solutions of Riccati's Equations.

Tabulated exact solution of $y'(x) + P(x)y(x) + Q(x)y\ (x) = R(x)$

	A particular solution	Required interrelationship
(i)	$a\, e^{-\int P dx}$	$R = a^2 Q\, e^{-2\int P dx}$
(ii)	$\dfrac{1}{\int Q dx + b}$	$R = \dfrac{P}{\int Q dx + b}$
(iii)	$\int R dx + c$	$Q = -\dfrac{P}{\int R dx + c}$
(iv)	$-\dfrac{P}{Q}$	$R = -\left(\dfrac{P}{Q}\right)'$
(v)	$\dfrac{R}{P}$	$Q = \left(\dfrac{P}{R}\right)'$
(vi)	$\sqrt{\dfrac{R}{Q}}$	$P = -\left(\ell n \sqrt{\dfrac{R}{Q}}\right)'$
(vii)	$\dfrac{e^{-\int(P-2Q)dx} - \int P e^{-\int(P-2Q)dx}\,dx - a}{e^{-\int(P-2Q)dx} + \int P e^{-\int(P-2Q)dx}\,dx + a}$	$R = Q - P$
(viii)	$\dfrac{\int P e^{-\int(P+2Q)dx}\,dx + b - e^{-\int(P+2Q)dx}}{\int P e^{-\int(P+2Q)dx}\,dx + b + e^{-\int(P+2Q)dx}}$	$R = Q + P$

NOTE: a, b, c are arbitrary constants.

These eight table entries represent a great number of specialized cases.

References

1. W. H. Louisell, "Coupled Mode and Parametric Electronics", John Wiley & Sons, 1960.

2. C. W. Burnes, "Conservative Coupling Between Modes of Propagation - a Tabular Summary", Proceedings of the IEEE, Vol. 52, No. 1, Jan. 1964, pp 64-73.

3. H. C. Huang and D. Y. Fan, "Coupled-Mode Theory of Guided Wave Propagation through a Ferrite Rod with Variable Cross-Section", Scientia Sinica, Vol. XIV, No. 3, Mar., 1965, pp 379-395.

4. S. E. Miller, "Coupled Wave Theory and Waveguide Applications", The Bell System Technical Journal, Vol. XXXIII, No. 3, May, 1954.

5. W. H. Louisell, "Analysis of the Single Tapered Mode Coupler", The Bell System Technical Journal, Vol. XXXIV, No. 4, July, 1955.

6. J. S. Cook, "Tapered-Velocity Coupler", The Bell System Technical Journal Vol. XXXIV, No. 4, July, 1955.

7. H. C. Huang, "Generalized Theory of Coupled Local Normal Modes in Multi-Wave Guides", Scientia Sinica, Vol. IX, No. 1, Jan., 1960, pp 142-154.

8. R. A. Sigelmann and D. K. Reynolds, "Traveling Wave Antenna with Broad Bandwidth", T.R. No. 42, University of Washington, Seattle, June, 1962.

9. G. N. Watson, "Theory of Bessel Functions", (2nd Ed.) London: Cambridge University Press, 1944.

10. G. M. Murphy, "Ordinary Differential Equations and Their Solutions", D. Van Nostrand, 1960.

11. P. Hartman, "Ordinary Differential Equations", John Wiley and Sons, 1964.

12. Iwao Sugai, "A Table of Solutions of Riccati Equations", Proc. IRE (Correspondence), Vol. 50, No. 10, Oct., 1962, pp 2125-2126.

13. I. Sugai, "Riccati's Nonlinear Differential Equation", Am. Math. Monthly, Vol. 67, pp 134-139, Feb., 1960.

14. I. Sugai, "Exact Solutions for Ordinary Nonlinear Differential Equations", Electrical Communication, Vol. 37, No. 2, pp 47-55, 1961.

15. A. K. Kajagopal, "On the Generalized Riccati Equation", Am. Math. Monthly, Vol. 68, pp 777-779, Oct., 1961.

16. I Sugai, "A Class of Solved Riccati's Equations", Electrical Communication, Vol. 37, No. 1, pp 56-60, 1961.

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13. ABSTRACT

In this report a systematic mathematical method is introduced for the solution of problems involving two coupled modes in a coupled system with varying parameters. These problems involve systems of linear differential equations with varying coefficients.

By the use of a linear transformation of the dependent variables and a double diagonalization process, the coupled mode equations are reduced to two decoupled Riccati equations. The final form of the general solution is obtained in terms of four varying coupling coefficients and a transform parameter.

To illustrate some applications of the method, solutions of two special cases which have been solved previously by other workers are obtained. The solutions for a number of special cases, in which the varying coefficients are specified or interrelated, are also obtained. Further possible applications are indicated.

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14. KEY WORDS	LINK A		LINK B		LINK C	
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Coupled modes						
Varying parameters						
Linear transformation						
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